

Moduli space volume of vortex and localization

Akiko Miyake^a, Kazutoshi Ohta^b and Norisuke Sakai^c

^aDepartment of General Education, Kushiro National College of Technology,
Kushiro 084-0916, Japan,

^bInstitute of Physics, Meiji Gakuin University, Yokohama 244-8539, Japan,

^cDepartment of Mathematics, Tokyo Woman's Christian University, Tokyo 167-8585, Japan

E-mail: ^amiyake@ippan.kushiro-ct.ac.jp; ^bkohta@law.meijigakuin.ac.jp;

^c Speaker, sakai@lab.twcu.ac.jp

Abstract. Volume of moduli space of BPS vortices on a compact genus h Riemann surface Σ_h is evaluated by means of topological field theory and localization technique. Vortex in Abelian gauge theory with a single charged scalar field (ANO vortex) is studied first and is found that the volume of the moduli space agrees with the previous results obtained more directly by integrating over the moduli space metric. Next we extend the evaluation to non-Abelian gauge groups and multi-flavors of scalar fields in the fundamental representation. We find that the result of localization can be consistently understood in terms of moduli matrix formalism wherever possible. More details are found in our paper[1].

1. Introduction

Static solitons exert no force between them at the critical coupling, and are called BPS solitons[2]. Since the BPS solitons can coexist at arbitrary positions, the solutions of the BPS equations have many parameters such as the position of the soliton, which are called moduli. The moduli space of the BPS solitons plays important roles in understanding dynamics of solitons. For instance, the thermodynamical partition function can be obtained from the volume of the moduli space, since the solitons behave as free particles on their moduli space, when they move slowly[3, 4]. Another non-trivial and important application of the volume of the moduli space is to obtain the nonperturbative effects, first found in the case of instantons. Nekrasov pointed out that the volume of the moduli space of the instantons, which is the so-called Nekrasov partition function, gives the non-perturbative effective prepotential of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory[5]. To obtain finite values of the volume of the moduli space, we consider vortices on compact base manifolds such as genus h Riemann surfaces. In recent years, BPS vortices in non-Abelian gauge theories have attracted much attention.[6, 7, 8, 9, 23, 11, 12, 13, 14, 15] The asymptotic metric of the moduli space is obtained for well-separated of non-Abelian vortices.[16] However, it is not enough to obtain the volume of the moduli space of the non-Abelian vortex.

In another approach, the “localization” technique[17, 18] of topological field theory has been applied successfully to evaluate the volume of the moduli space[19, 20]. The Kähler structure of moduli space induces the localization property in the integration of the Kähler form which gives the volumes. The localization means that the integral of the volume form over the moduli space is localized (dominated) at isometry fixed points of the moduli space. This localization simplifies the evaluation of the volume of the moduli space drastically. The localization technique is based

on a topological field theory, which can be understood as a twisted version of the supersymmetric theories[17].

We apply the localization technique in the evaluation of the moduli space of the BPS vortices on Riemann surfaces in Abelian as well as non-Abelian $U(N_c)$ gauge theories. We consider N_f flavors of Higgs fields in the fundamental representation extending the results in Ref. [4, 21] to the case of multiflavors, in the Abelian as well as the non-Abelian gauge theories. We find that our results by the localization technique completely agree with the previous results[4, 21] for any topology of the Riemann surface for the ANO vortices ($N_c = N_f = 1$). We can also evaluate the volume of the moduli space of the non-Abelian vortices, although it has been difficult to construct the metric of moduli space of the non-Abelian vortex apart from well-separated local vortices ($N_f = N_c$) [16]. By imposing the BPS equations as constraints in the field configuration space, we can regard the moduli space of BPS solitons as the quotient space of the fields, similarly to the Kähler quotient space. The integrals to give the volume of the moduli space reduces to simple residue integrals in the localization technique. Consequently we can evaluate the volume of the moduli space of the BPS vortices much easier than the explicit construction of the metric from the BPS solutions. We also work out the metric of the moduli space of single Abelian as well as non-Abelian vortices using moduli matrix formalism[9] in order to compare it to our result from localization technique.

2. BPS vortices on Riemann surfaces

Let us consider a $(2 + 1)$ -dimensional space-time with the line element

$$ds^2 = -dt^2 + \sigma[(dx)^2 + (dy)^2] = -dt^2 + g_{z\bar{z}}dzd\bar{z} = g_{\mu\nu}dx^\mu dx^\nu, \quad (1)$$

where the conformal factor and the complex coordinate are denoted as $\sigma = g_{z\bar{z}}$ and $z = x + iy$, respectively. We will denote the time coordinate t and the spacial coordinates x, y by 0 and $i, j = 1, 2$, respectively, and space-time coordinates by $\mu, \nu = 0, 1, 2$. We also define the Kähler 2-form $\omega = \frac{i}{2}g_{z\bar{z}}dz \wedge d\bar{z}$ from the metric. Then the area \mathcal{A} of the Riemann surface Σ_h is given by $\mathcal{A} = \int dxdy \sigma = \int_{\Sigma_h} \omega$.

We are interested in a $U(N_c)$ gauge theory in $(2 + 1)$ -dimensional space-time with gauge fields A_μ as $N_c \times N_c$ matrices and N_f Higgs fields in the fundamental representation of the $SU(N_c)_C$ H as an $N_c \times N_f$ matrix. The Lagrangian is given in with the gauge coupling g and the Fayet-Iliopoulos (FI) parameter c as

$$L = \int dxdy \sqrt{-\det(g_{\mu\nu})} \mathcal{L} = \int dt dxdy \sigma \mathcal{L} \quad (2)$$

$$\mathcal{L} = \text{Tr} \left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{g^2}{4} (HH^\dagger - c \mathbf{1}_{N_c})^2 \right], \quad (3)$$

where the covariant derivative and the field strength are defined by $D_\mu \equiv \partial_\mu - iA_\mu$ and $F_{\mu\nu} = i[D_\mu, D_\nu]$. This Lagrangian can be embedded into a supersymmetric theory with eight supercharges.

The Bogomol'nyi bound for the energy E of vortices (depending on x, y only) is obtained

$$\begin{aligned} E &= \int dxdy \text{Tr} \left[4D_{\bar{z}} H D_z H^\dagger + \frac{1}{g^2 \sigma} \left(F_{12} - \frac{g^2 \sigma}{2} (c - HH^\dagger) \right)^2 + c F_{12} \right] \\ &\geq c \int dxdy \text{Tr}(F_{12}) = 2\pi c k, \end{aligned} \quad (4)$$

where is the topological charge k is the vorticity, namely the number of vortices. The bound is saturated if the following BPS equations

$$\mu_r = F - \frac{g^2}{2}(c - HH^\dagger)\omega, \quad (5)$$

$$\mu_{\bar{z}} = \mathcal{D}_{\bar{z}}H, \quad \mu_z = \mathcal{D}_zH^\dagger, \quad (6)$$

are satisfied. The Abelian gauge theory corresponds to $N_c = 1$. Using these moment maps, the moduli space \mathcal{M}_k of the vortex with the vorticity k is given by a Kähler quotient space

3. Abelian vortices

Let us consider the volume of the moduli space of BPS vortices in Abelian ($G = U(1)$) gauge theory with N_f Higgs fields on a compact Riemann surface Σ_h of genus h . The Higgs field $H(z, \bar{z})$ has unit charge and is represented by an N_f component vector.

We now introduce fermionic fields λ, ψ to define BRST transformations for fields as follows

$$\begin{aligned} QA &= i\lambda, & Q\lambda &= -d\Phi, \\ QH &= i\psi, & Q\psi &= \Phi H, & QH^\dagger &= -i\psi^\dagger, & Q\psi^\dagger &= \Phi H^\dagger, \\ QY &= \Phi * \chi, & Q\chi &= Y, & Q\Phi &= 0, \end{aligned} \quad (7)$$

where we have used form notations. These BRST transformations are nilpotent up to gauge transformations, namely $Q^2 = -i\delta_\Phi$, where δ_Φ is the generator of the gauge transformation with infinitesimal parameter Φ . Thus if we consider gauge invariant operators only, the BRST transformation Q forms a cohomology for those operators, which is called the “equivariant cohomology”. The equivariant cohomology clarifies topological aspects of (topological) field theory considering, and will play an essential but indirect role of the “localization” in the evaluation of the volume.

Φ is BRST closed itself, so any function $\mathcal{O}_0 \equiv \mathcal{W}(\Phi)$ is also BRST closed. In the sense of the BRST symmetry, the 0-form operator becomes a good (physical) observable. The 0-form observable satisfies the so-called descent relation

$$d\mathcal{O}_0 + Q\mathcal{O}_1 = 0, \quad \mathcal{O}_1 \equiv \frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} \lambda. \quad (8)$$

This fact means that the integral of \mathcal{O}_1 along a closed circle γ on Σ_h $\int_\gamma \mathcal{O}_1$ is BRST closed and a good cohomological observable. Similarly, \mathcal{O}_1 satisfies

$$d\mathcal{O}_1 + Q\mathcal{O}_2 = 0, \quad \mathcal{O}_2 \equiv i \frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} F + \frac{1}{2} \frac{\partial^2 \mathcal{W}(\Phi)}{\partial \Phi^2} \lambda \wedge \lambda. \quad (9)$$

Thus the operator $\int_{\Sigma_h} \mathcal{O}_2$ also becomes BRST closed. If we choose $\mathcal{W}(\Phi) = \frac{1}{2}\Phi^2$, we find that the integral $\int_{\Sigma_h} [i\Phi F + \frac{1}{2}\lambda \wedge \lambda]$ is BRST closed. Furthermore, we can see an integral $\int_{\Sigma_h} [i\Phi HH^\dagger \omega + \psi^\dagger \psi \omega]$ is also BRST closed, since the integrand can be written by the BRST exact form $Q[-iH\psi^\dagger \omega]$. Therefore we obtain a BRST action

$$S_0 = \int_{\Sigma_h} \left[i\Phi \mu_r + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega \right] \quad (10)$$

as a BRST invariant completion of the BPS constraint (5), since Φ can be regarded as a Lagrange multiplier. We can implement the remaining BPS constraint (6) by introducing the BRST exact term S_1 to the action

$$S = S_0 + S_1, \quad (11)$$

$$S_1 = t_1 Q \int_{\Sigma_h} i\chi \wedge * \mu_c, \quad (12)$$

where $\mu_c \equiv \mu_z dz + \mu_{\bar{z}} d\bar{z}$. The volume of the moduli space of the BPS equations (5) and (6) is obtained from the following integral

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 Y \mathcal{D}^2 \chi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi e^{-S}, \quad (13)$$

where the path integral is to satisfy the constraint $\frac{1}{2\pi} \int F = k$.

If the action includes BRST exact terms with couplings, the path integral does not depend on the couplings. Using this coupling independence of the integral, we can add the following BRST exact term to the action S in Eq. (11) without changing the value of the integral

$$S_2 = t_2 Q \int_{\Sigma_h} i\chi \wedge *Y. \quad (14)$$

This part of the action serves to impose the remaining BPS constraint (6). By exploiting the coupling independence, we can go to a parameter region where the integral can be easily performed: let us take the limit $t_1 \rightarrow 0$ and $t_2 \rightarrow 1$ of the BRST exact terms. Then the action becomes

$$\begin{aligned} S' &= \int_{\Sigma_h} \left[i\Phi \mu_r + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega \right] + Q \int_{\Sigma_h} i\chi \wedge *Y \\ &= \int_{\Sigma_h} \left[i\Phi \left\{ F - \frac{g^2}{2} (c - HH^\dagger) \omega \right\} + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega + iY \wedge *Y + i\Phi \chi \wedge \chi \right]. \end{aligned} \quad (15)$$

Thus we can use the integral

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 Y \mathcal{D}^2 \chi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi e^{-S'}, \quad (16)$$

to evaluate the volume of moduli space \mathcal{M}_k .

First of all, we wish to integrate out the matter fields H , ψ , Y and χ , whose integrals are Gaussian. Neglecting the possible anomalies coming from the fermionic zero modes of matter fields, we obtain

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda (i\Phi)^{N_f(\dim \Omega^1 \otimes \mathcal{L}_k - \dim \Omega^0 \otimes \mathcal{L}_k)} e^{-\int_{\Sigma_h} [i\Phi(F - \frac{g^2}{2} \omega) + \frac{1}{2} \lambda \wedge \lambda]}, \quad (17)$$

where $\dim \Omega^n \otimes \mathcal{L}_k$ ($n = 0, 1$) stands for the number of holomorphic n -forms coupled with $U(1)$ gauge field (holomorphic line bundle) of the topological charge k . The Hirzebruch-Riemann-Roch theorem says

$$\dim \Omega^0 \otimes \mathcal{L}_k - \dim \Omega^1 \otimes \mathcal{L}_k = 1 - h + \frac{1}{2\pi} \int_{\Sigma_h} F = 1 - h + k. \quad (18)$$

Thus we have

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda \frac{1}{(i\Phi)^{N_f(1-h+k)}} e^{-\int_{\Sigma_h} [i\Phi(F - \frac{g^2}{2} \omega) + \frac{1}{2} \lambda \wedge \lambda]}. \quad (19)$$

By using $2(1-h) = \frac{1}{4\pi} \int_{\Sigma_h} R^{(2)}$ and $k = \frac{1}{2\pi} \int_{\Sigma_h} F$ in terms of the curvature 2-form $R^{(2)}$ of the Riemann surface and the field strength 2-form F , we can exponentiate powers of Φ in Eq. (19) to obtain

$$\mathcal{V}_k = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda e^{-S_{\text{eff}}}, \quad S_{\text{eff}} = S_R + S_F + S_V, \quad (20)$$

where

$$S_R = \frac{1}{8\pi} \int_{\Sigma_h} \log(i\Phi) R^{(2)}, \quad S_V = -i \frac{g^2 c}{2} \int_{\Sigma_h} \Phi \omega, \quad (21)$$

$$S_F = \int_{\Sigma_h} \left[i \left(\Phi + \frac{1}{2\pi i} \log i\Phi \right) F + \frac{1}{2} \lambda \wedge \lambda \right]. \quad (22)$$

However S_F is not invariant under the BRST symmetry (not BRST closed). Since any regularization scheme should preserve the BRST symmetry, this means that we have overlooked contributions from the fermionic zero modes in the integrals of fields ψ, χ . To recover the contributions from the fermionic zero modes, we notice that the BRST closed action must take the form (9) given by the descent relation

$$S'_F = \int_{\Sigma_h} \left[i \frac{\partial \mathcal{W}_{\text{eff}}}{\partial \Phi} F + \frac{1}{2} \frac{\partial^2 \mathcal{W}_{\text{eff}}}{\partial \Phi^2} \lambda \wedge \lambda \right], \quad (23)$$

with $\mathcal{W}_{\text{eff}}(\Phi) = \frac{1}{2} \Phi^2 + \frac{N_f}{2\pi i} \Phi (\log i\Phi - 1)$. Then we obtain

$$S'_F = \int_{\Sigma_h} \left[i \left(\Phi + \frac{1}{2\pi i} \log i\Phi \right) F + \frac{\mu(\Phi)}{2} \lambda \wedge \lambda \right], \quad \mu(\Phi) = \frac{\partial^2 \mathcal{W}_{\text{eff}}}{\partial \Phi^2} = 1 + \frac{N_f}{2\pi i \Phi}. \quad (24)$$

The only correction due to (previously neglected) anomalies of the fermionic zero modes is changing the coefficient of $\frac{1}{2} \lambda \wedge \lambda$ from unity to $\mu(\Phi)$, which assures the BRST symmetry of the effective action.

Using $\int_{\Sigma_h} \omega = \mathcal{A}$ and $\int_{\Sigma_h} F = 2\pi k$, we integrate over A and λ and finally reduce the path integral into the following one-dimensional integral over ordinary real number ϕ (the constant mode of the field Φ)

$$\mathcal{V}_k = \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \frac{\mu(\phi)^h}{(i\phi)^{N_f(1-h+k)}} e^{i\phi \left(\frac{g^2 c}{2} \mathcal{A} - 2\pi k \right)}, \quad (25)$$

where $\mu(\phi)$ is defined in Eq. (24). Now let us evaluate the above integral. Since the integrand has a pole at $\phi = 0$, we need to look for the correct integration contour to avoid the pole. The term $\int i\Phi H H^\dagger \omega$ in the action (15) of the path integral reveals that we need to choose the contour below the real axis in order to assure the convergence of path integral of matter fields H . Namely we should avoid the pole at $\phi = 0$ counter-clock-wise below the pole. Expanding the integrand in powers of ϕ , we can integrate term by term. We find that the volume is nonvanishing only if

$$\frac{g^2 c}{2} \mathcal{A} - 2\pi k \geq 0, \quad (26)$$

and

$$\mathcal{V}_k = \sum_{j=0}^{\min(h,d)} \frac{h!}{j!(h-j)!} \left(\frac{N_f}{2\pi} \right)^{h-j} \frac{1}{(d-j)!} \left(\frac{g^2 c}{2} \mathcal{A} - 2\pi k \right)^{d-j}, \quad (27)$$

where we have defined $d \equiv kN_f + (1-h)(N_f-1)$ and used the residue integral formula. Eq.(26) means that there is no solution for $\mathcal{A} < \frac{4\pi}{g^2 c} k$. This result is in agreement with the bound found in the case of ANO vortices ($N_c = N_f = 1$), which is known as the Bradlow bound[22]. More interestingly, the non-vanishing volume exists only if $d \geq 0$, namely $k \geq (h-1) \frac{N_f-1}{N_f}$. So we can

choose any non-negative k for $h = 0, 1$ or $N_f = 1$, but k must be sufficiently large for $h > 1$ in the case of $N_f > 1$.

Let us consider the simplest case $N_f = 1$, namely the ANO vortices. In this case, Eq. (27) reduces to

$$\mathcal{V}_k = \sum_{j=0}^{\min(h,k)} \frac{h!}{j!(k-j)!(h-j)!} \left(\frac{g^2 c}{2} \mathcal{A} - 2\pi k \right)^{k-j}. \quad (28)$$

To single out the net contribution from the k vortex sector, we can mod out the contribution from the vacuum to define $\tilde{\mathcal{V}}_k \equiv \mathcal{V}_k / \mathcal{V}_0$. We find that our result agrees exactly with the previous result of the volume of the moduli space obtained from the moduli space metric by Manton and Nasir[21], apart from an intrinsically ambiguous normalization constant to define the moduli space metric of a single vortex moduli space.

For semi-local vortices on sphere, we find the volume of the moduli space of vortices as

$$\mathcal{V}_k(S^2) = \frac{1}{(kN_f + N_f - 1)!} \left(\frac{g^2 c}{2} \mathcal{A} - 2\pi k \right)^{kN_f + N_f - 1}. \quad (29)$$

4. Non-Abelian Vortex

Similarly to the Abelian case, the BRST transformations for non-Abelian case is given by

$$\begin{aligned} QA &= i\lambda, & Q\lambda &= -d_A\Phi, & Q\Phi &= 0, \\ QH &= i\psi, & Q\psi &= \Phi H, & QH^\dagger &= -i\psi^\dagger, & Q\psi^\dagger &= H^\dagger\Phi, \\ QY_z &= i\Phi\chi_z & Q\chi_z &= Y_z, & QY_{\bar{z}} &= -i\chi_{\bar{z}}\Phi, & Q\chi_{\bar{z}} &= Y_{\bar{z}}, \end{aligned} \quad (30)$$

where $d_A\Phi \equiv d\Phi - i[A, \Phi]$. The volume of non-Abelian vortices can be obtained by evaluating the following path integral

$$\begin{aligned} S &= S_0 + t_1 S_1 + t_2 S_2, \\ S_0 &= \int_{\Sigma_h} \text{Tr} \left[i\Phi \left\{ F - \frac{g^2}{2} (c - HH^\dagger) \omega \right\} + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega \right], \\ S_1 &= Q \int_{\Sigma_h} d^2 z \text{Tr} \left[\frac{1}{2} g^{z\bar{z}} (\chi_z \mu_{\bar{z}} + \mu_z \chi_{\bar{z}}) \right], \quad S_2 = Q \int_{\Sigma_h} d^2 z \text{Tr} \left[\frac{1}{2} g^{z\bar{z}} (\chi_z Y_{\bar{z}} + Y_z \chi_{\bar{z}}) \right]. \end{aligned} \quad (31)$$

To evaluate the volume, it is easiest to choose the parameters $t_1 = 0, t_2 = 1$. We obtain

$$\mathcal{V}_k^{N_c, N_f}(\Sigma_h) = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi \mathcal{D}^2 Y \mathcal{D}^2 \chi e^{-S_0 - S_2}. \quad (32)$$

To integrate out the fields, we choose a gauge which diagonalizes Φ as $\Phi = \text{diag}(\phi_1, \phi_2, \dots, \phi_{N_c})$. After integrating out H, ψ, Y, χ and off-diagonal pieces of A and λ together with the associated ghosts, the integral reduces to the $U(1)^{N_c}$ gauge theory

$$\begin{aligned} \mathcal{V}_k^{N_c, N_f} &= \int \prod_{a=1}^{N_c} (\mathcal{D}\phi_a \mathcal{D}^2 A_a \mathcal{D}^2 \lambda_a) \frac{\prod_{a \neq b} (i\phi_a - i\phi_b)^{\dim \Omega^0 \otimes \mathcal{L}_{k_a} \otimes \mathcal{L}_{k_b}^{-1} - \dim \Omega^1 \otimes \mathcal{L}_{k_a} \otimes \mathcal{L}_{k_b}^{-1}}}{\prod_{a=1}^{N_c} (i\phi_a)^{N_f (\dim \Omega^0 \otimes \mathcal{L}_{k_a} - \dim \Omega^1 \otimes \mathcal{L}_{k_a})}} \\ &\quad \times e^{-\sum_{a=1}^{N_c} \int_{\Sigma_h} [i\phi_a (F^{(a)} - \frac{g^2 c}{2} \omega) + \frac{1}{2} \lambda_a \wedge \lambda_a]}, \end{aligned} \quad (33)$$

where the diagonal a -th $U(1)$ gauge field, field strength and gaugino are denoted as $A_a, F^{(a)}$ and λ_a , and k_a 's are diagonal $U(1)$ topological charges $\frac{1}{2\pi} \int F^{(a)} = k_a$, which satisfies the constraint of the total topological charge $k = \sum_{a=1}^{N_c} k_a$.

Similarly to the Abelian case, after using the Hirzebruch-Riemann-Roch theorem and supplementing the effective action to satisfy the BRST invariance, we can integrate over A_a and λ_a to obtain the following finite dimensional integral of zero mode ϕ_a of Φ_a

$$\mathcal{V}_k^{N_c, N_f} = \sum_{\sum_a k_a = k} (-1)^\sigma \int \prod_{a=1}^{N_c} \frac{d\phi_a}{2\pi} \frac{\mu(\phi)^h \prod_{a < b} (i\phi_a - i\phi_b)^{2-2h}}{\prod_{a=1}^{N_c} (i\phi_a)^{N_f(1-h+k_a)}} e^{2\pi i \sum_{a=1}^{N_c} \phi_a (\mathcal{A} - k_a)}, \quad (34)$$

where $\tilde{\mathcal{A}} = \frac{g^2 c}{4\pi} \mathcal{A}$, $\mu(\phi) = \prod_{a=1}^{N_c} \left(1 + \frac{1}{2\pi i} \frac{N_f}{\phi_a}\right)$, and $\sigma = \frac{1}{2} N_c (N_c - 1) (1 - h) - \sum_{a < b} (k_a - k_b)$.

We will consider the sphere topology for the Riemann surface. Let us first examine the result for $N_c = 2$. For semi-local vortices ($N_f > N_c = 2$), we find the asymptotic power of \mathcal{A} for large area $\mathcal{A} \rightarrow \infty$ as

$$\mathcal{V}_k^{2, N_f}(S^2) \propto \tilde{\mathcal{A}}^{k N_f + 2(N_f - 2)}. \quad (35)$$

In contrast, the asymptotic power is much smaller for local vortices ($N_f = N_c = 2$)

$$\mathcal{V}_k^{2, 2}(S^2) = \frac{2(2\pi)^{2k} \tilde{\mathcal{A}}^k}{k!} + \mathcal{O}(\tilde{\mathcal{A}}^{k-1}). \quad (36)$$

Precise values of the volume of local vortices are given by

$$\begin{aligned} \mathcal{V}_0^{2, 2}(S^2) &= 2, \\ \mathcal{V}_1^{2, 2}(S^2) &= 2(2\pi)^2 (\tilde{\mathcal{A}} - 1), \\ \mathcal{V}_2^{2, 2}(S^2) &= \frac{2(2\pi)^4}{2!} \left(\tilde{\mathcal{A}}^2 - \frac{20}{6} \tilde{\mathcal{A}} + \frac{17}{6} \right), \\ \mathcal{V}_3^{2, 2}(S^2) &= \frac{2(2\pi)^6}{3!} \left(\tilde{\mathcal{A}}^3 - 7\tilde{\mathcal{A}}^2 + \frac{331}{20} \tilde{\mathcal{A}} - \frac{793}{60} \right), \\ \mathcal{V}_4^{2, 2}(S^2) &= \frac{2(2\pi)^8}{4!} \left(\tilde{\mathcal{A}}^4 - 12\tilde{\mathcal{A}}^3 + \frac{818}{15} \tilde{\mathcal{A}}^2 - \frac{2336}{21} \tilde{\mathcal{A}} + \frac{18047}{210} \right). \end{aligned} \quad (37)$$

Let us next consider $N_c = 3$ case. The asymptotic power of \mathcal{A} for semi-local vortices ($N_f > N_c = 3$) is given by

$$\mathcal{V}_k^{N_c=3, N_f}(S^2) \propto \tilde{\mathcal{A}}^{k N_f + 3(N_f - 3)}. \quad (38)$$

Combining the $N_c = 2, 3$ results (35) and (38) of the asymptotic power of \mathcal{A} for semi-local vortices ($N_f > N_c$), we conjecture the asymptotic power for generic semi-local vortices $N_f > N_c$ as

$$\mathcal{V}_k^{N_c, N_f}(S^2) \propto \tilde{\mathcal{A}}^{k N_f + N_c(N_f - N_c)}. \quad (39)$$

Similarly to the $N_c = 2$ case, we observe the reduction of the asymptotic power of \mathcal{A} for local vortices ($N_f = N_c$) compared to the semi-local vortices

$$\mathcal{V}_k^{3, 3}(S^2) = \frac{3!}{k!} \left(\frac{(2\pi)^3 \tilde{\mathcal{A}}}{2} \right)^k + \mathcal{O}(\tilde{\mathcal{A}}^{k-1}). \quad (40)$$

Combining the results (36) and (40) of $N_c = 2, 3$ cases, we conjecture the asymptotic power of \mathcal{A} of the volume of k local vortices for the general values of $N_c = N_f$ as

$$\mathcal{V}_k^{N, N}(S^2) = \frac{N!}{k!} \left(\frac{(2\pi)^N \tilde{\mathcal{A}}}{N-1} \right)^k + \mathcal{O}(\tilde{\mathcal{A}}^{k-1}). \quad (41)$$

In the case of the local Vortex $N_c = N_f = 3$,

$$\begin{aligned}
\mathcal{V}_0^{3,3}(S^2) &= 3!, \\
\mathcal{V}_1^{3,3}(S^2) &= 3! \times \frac{(2\pi)^3}{2}(\tilde{\mathcal{A}} - 1), \\
\mathcal{V}_2^{3,3}(S^2) &= \frac{3!}{2!} \times \frac{(2\pi)^6}{2^2} \left(\tilde{\mathcal{A}}^2 - \frac{46}{15}\tilde{\mathcal{A}} + \frac{36}{15} \right), \\
\mathcal{V}_3^{3,3}(S^2) &= \frac{3!}{3!} \times \frac{(2\pi)^9}{2^3} \left(\tilde{\mathcal{A}}^3 - \frac{31}{5}\tilde{\mathcal{A}}^2 + \frac{3641}{280}\tilde{\mathcal{A}} - \frac{23249}{2520} \right).
\end{aligned} \tag{42}$$

We have also computed the case of torus topology for the base manifold. More detailed results can be found in Ref. [1].

5. Effective Lagrangian of Vortices

In order to understand the reduction of the asymptotic power of \mathcal{A} of local vortices ($N_f = N_c$) compared to semi-local vortices ($N_f > N_c$), we study the effective Lagrangian by using the moduli matrix formalism[9].

For arbitrary vortex number, we first work out the effective Lagrangian of Abelian semi-local vortices ($N_f > N_c = 1$) on the sphere, using the strong coupling $g^2 \rightarrow \infty$. In the strong coupling limit, the solution Ω of the master equation is solved algebraically $\Omega = H_0 H_0^\dagger / c$. Therefore the boundary condition for the vortex number k is satisfied by requiring that at least one of components of moduli matrix to be a polynomial of order k , and all other components to be at most of order k

$$H_0^{(k)}(z) = \sqrt{c} \left(\sum_{j=0}^k a_j^{(1)} z^j, \dots, \sum_{j=0}^k a_j^{(N_f)} z^j \right), \tag{43}$$

where at least one of the coefficients of the k -th power is non-vanishing: $a_k^{(j)} \neq 0$. Compared to the usual noncompact plane, we emphasize two new features of the vortex moduli on the compact Riemann surfaces which are realized in the moduli matrix (43). Firstly we allow the leading power of z to be in any components. If we use global $SU(N_f)$ rotations combined with the V -transformations, it is possible to place the leading power to be in a particular component, say in the first component. This form is the usual choice for the moduli matrix on noncompact flat plane [11]. These new $N_f - 1$ moduli may be regarded as an orientation of the vacuum at infinity and are nonnormalizable on noncompact plane: $(a_k^{(1)}, \dots, a_k^{(N_f)})/a_k^{(1)}$ after dividing out by V -transformations. These $N_f - 1$ extra complex moduli parameters are present even in the case of vacuum ($k = 0$) on compact Riemann surfaces. Secondly the additional $N_f - 1$ “size” moduli are retained on compact Riemann surfaces, since they become normalizable and are dynamical variables in the effective Lagrangian. More specifically, the standard moduli matrix on noncompact plane contains up to only $(k - 1)$ -th power of z except in the first component. The $N_f - 1$ coefficients of these $(k - 1)$ -th power represent “size” of vortices, are nonnormalizable, and have to be fixed by the boundary condition on noncompact plane. Both of the “vacuum” and the “size” moduli become normalizable and provide additional $2N_f - 2$ complex moduli on compact Riemann surfaces. Let us emphasize that these new moduli should be fixed by boundary conditions and will not be a genuine moduli parameter of the solution once we take the limit of noncompact base manifold, such as a plane.

The Kähler potential of the effective Lagrangian has been obtained[23]. We generalize the formula to the case of curved manifold such as Riemann surfaces, and insert the moduli matrix

(43) to obtain the Kähler potential on the sphere

$$K^{(k)} = \mathcal{A}c \int_{S^2} dx dy \frac{1}{\pi(1+|z|^2)} \log \left(\sum_{i=1}^{N_f} \left| \sum_{j=0}^k a_j^{(i)} z^j \right|^2 \right). \quad (44)$$

We find that the integral is convergent and is proportional to \mathcal{A} , indicating that all the moduli parameters take values of order \mathcal{A} . So we find that each complex moduli gives a power of \mathcal{A} . Since there are $kN_f + N_f - 1$ complex moduli, we obtain the volume of the moduli space asymptotically $\mathcal{A} \rightarrow \infty$ to be proportional to the $kN_f + N_f - 1$ power of \mathcal{A}

$$\hat{\mathcal{V}}_k^{1,N_f}(S^2) = (Nc\pi\mathcal{A})^{kN_f+N_f-1} \times \int \prod_{i=1}^{N_f} \prod_{j=0}^k da_j^{(i)} \prod_{i'=1}^{N_f} \prod_{j'=0}^k d\bar{a}_{j'}^{(i')} \frac{\partial}{\partial a_j^{(i)}} \frac{\partial}{\partial \bar{a}_{j'}^{(i')}} \left(\frac{K^{(k)}}{c\pi\mathcal{A}} \right), \quad (45)$$

where the coefficient of $(Nc\pi\mathcal{A})^{kN_f+N_f-1}$ is given by an integral representation for $K^{(k)}$. This asymptotic power agrees with the result (29) of the topological field theory.

We have also worked out the effective Lagrangian on sphere for non-Abelian semi-local vortices ($N_f > N_c > 1$) [1] by using the moduli matrix approach[9]. Similarly to the Abelian semi-local vortices, we have found that the each complex moduli contributes to an asymptotic power of \mathcal{A} at the strong coupling limit, in agreement with the result (39) of the topological field theory.

In the case of non-Abelian local vortices with $N_f = N_c = N$, we have worked out previously the metric of a single vortex on a plane in the moduli matrix formalism[16]. We found that only the position moduli can be of order $\sqrt{\mathcal{A}}$, whereas other orientational moduli consists of $\mathbb{C}P^{N-1}$ with the radius of order $1/g\sqrt{c}$. Moduli space of multi-vortices $k > 1$ are symmetric product of k moduli spaces of each single vortex except for separations of order smaller than the vortex size $1/g\sqrt{c}$. These facts imply that the orientational moduli can only give a finite volume unrelated to \mathcal{A} , whereas the vortex position can be of order $\sqrt{\mathcal{A}}$. Therefore the volume of the moduli space for k local non-Abelian vortices is proportional to \mathcal{A}^k , which agrees with our result (41) of the topological field theory.

6. Conclusion

The volume of moduli space of vortices are computed for $U(N_c)$ gauge theory with N_f Higgs fields in the fundamental representation, using the localization technique of topological field theory.

Volume of moduli space of ANO vortices ($N_f = N_c = 1$) for any vortex number k and for any genus h of Riemann surfaces is obtained and agrees with the previous direct calculation using the effective Lagrangian.

Volume of moduli space is obtained both for Abelian semi-local vortices ($N_f > N_c = 1$) and non-Abelian vortices ($N_c > 1$).

We find that the asymptotic power of area \mathcal{A} for $\mathcal{A} \rightarrow \infty$ is $\mathcal{A}^{kN_f+N_c(N_f-N_c)}$ for semi-local vortices ($N_f > N_c$), and that the power reduces to \mathcal{A}^k for local vortices ($N_f = N_c$).

Reduction of asymptotic power is understood by noticing that internal modes other than position do not extend over the whole area \mathcal{A} for local vortices.

Localization technique should be powerful to obtain the volume of moduli space of other solitons[24].

Acknowledgements

One of the authors (K.O.) would like to thank M. Nitta, K. Ohashi and Y. Yoshida for useful discussions and comments. One of the authors (N.S.) thanks Nick Manton, Norman Rink,

Makoto Sakamoto, and David Tong for a useful discussion. This work is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan No.19740120 (K.O.), No.21540279 (N.S.) and No.21244036 (N.S.), and by Japan Society for the Promotion of Science (JSPS) and Academy of Sciences of the Czech Republic (ASCR) under the Japan - Czech Republic Research Cooperative Program (N.S.).

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